

Universität des Saarlandes



Fachrichtung 6.1 – Mathematik

Preprint Nr. 163

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Saarbrücken 2005

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On the Hilbert-Samuel multiplicity of Fredholm tuples

Jörg Eschmeier¹

Abstract. For commuting tuples $R \in L(Z)^n$ of Banach-space operators that arise as quotients of lower semi-Fredholm systems $T \in L(X)^n$ with constant cohomology dimension $\dim H^n(z - T, X)$ near the origin $0 \in \mathbb{C}^n$, we show that the Hilbert-Samuel multiplicity of R calculates the rank of the cohomology sheaf $\mathcal{H}^n(z - R, \mathcal{O}_{\mathbb{C}^n}^Z)$ at $z = 0$.

0. Introduction

Let $T \in L(X)^n$ be a commuting tuple of bounded linear operators on a complex Banach space X , and let $K^\bullet(T, X)$ be the Koszul complex of T . The tuple T is said to be *lower semi-Fredholm* if the last cohomology group $H^n(T, X) = X / \sum_{i=1}^n T_i X$ of its Koszul complex is finite dimensional. In this case all the spaces $M_k(T) = \sum_{|\alpha|=k} T^\alpha X$ ($k \in \mathbb{N}$) are finite codimensional, and the direct sum $\oplus_{k \geq 0} M_k(T) / M_{k+1}(T)$ can be turned into a graded finitely generated $\mathbb{C}[z]$ -module. It is a fundamental result of commutative algebra that to any such module there is a polynomial $p \in \mathbb{Q}[x]$ of degree $\leq n$, the *Hilbert-Samuel polynomial*, with $\dim X / M_k(T) = p(k)$ for large k and such that the leading coefficient multiplied with $n!$ is a natural number, the so-called *Hilbert-Samuel multiplicity*.

On the other hand, for a given lower semi-Fredholm tuple $T \in L(X)^n$, there is an open neighbourhood U of $0 \in \mathbb{C}^n$ such that $\dim H^n(z - T, X) < \infty$ for all $z \in U$ and such that the last cohomology sheaf $\mathcal{H} = \mathcal{H}^n(z - T, \mathcal{O}_U^X)$ of the induced complex $K^\bullet(z - T, \mathcal{O}_U^X)$ of \mathcal{O}_U -modules is isomorphic to a quotient of a free module \mathcal{O}_U^N on U . In particular, the stalk \mathcal{H}_0 is a noetherian module over the local ring \mathcal{O}_0 of all convergent power series at $z = 0$ and hence possesses a Hilbert-Samuel polynomial $p_{an} \in \mathbb{Q}[x]$ such that $\dim \mathcal{H}_0 / \mathfrak{m}^k \mathcal{H}_0 = p_{an}(k)$ for large k . Here \mathfrak{m} is the maximal ideal of \mathcal{O}_0 .

Both versions of the Hilbert-Samuel polynomial and their leading coefficients were introduced by Douglas and Yan in [3]. In a series of papers Xiang Fang studied the properties of the Hilbert-Samuel multiplicity

$$c(T) = n! \lim_{k \rightarrow \infty} \frac{\dim X / M_k(T)}{k^n}$$

¹2000 Mathematics Subject Classification. Primary 47A15; Secondary 47A13, 32C35.

and its relations to other invariants in operator theory. In [6] a complete description of the relation between Fredholm index and the Hilbert-Samuel multiplicity for single operators is given, while [8] contains a variety of results showing the close connection between Hilbert-Samuel multiplicity and the Fredholm index in multivariable operator theory, including the observation that, for row contractions with finite defect, the Hilbert-Samuel multiplicity coincides with Arveson's curvature invariant [1].

In this note we extend a method originating in [7] to show that, for a suitable class of lower semi-Fredholm tuples $R \in L(Z)^n$ on a Banach space Z , the Hilbert-Samuel multiplicity coincides with the rank of the cohomology sheaf $\mathcal{H} = \mathcal{H}^n(z - R, \mathcal{O}_U^Z)$ at $z = 0$, or equivalently, that there are an open neighbourhood U of 0 and a proper analytic subset $S \subset U$ such that

$$c(R) = \min_{z \in U} \dim H^n(z - R, Z) = \dim H^n(w - R, Z)$$

for $w \in U \setminus S$. Furthermore, the restriction of \mathcal{H} to $U \setminus S$ is a locally free analytic sheaf of rank $c(R)$. These results are shown to hold for tuples $R \in L(Z)^n$ arising as quotients of lower semi-Fredholm tuples $T \in L(X)^n$ which are regular in the sense that the cohomology groups $H^n(z - T, X)$ have constant dimension near 0. Hence, in this setting, we prove that the Hilbert-Samuel multiplicity calculates the stabilized dimension of $H^n(z - R, Z)$ at $z = 0$, and give a natural explanation of results previously known in more particular situations.

1. Main results

Let $T = (T_1, \dots, T_n) \in L(X)^n$ be a commuting tuple of continuous linear operators on a complex Banach space X . The Koszul complex of T

$$K^\bullet(T, X) : 0 \rightarrow \Lambda^0 X \xrightarrow{T} \Lambda^1 X \xrightarrow{T} \dots \xrightarrow{T} \Lambda^n X \rightarrow 0$$

is a finite complex of bounded operators between Banach spaces (see §2.2 in [4]). The vector spaces

$$H^p(T, X) = \text{Ker}(\Lambda^p X \xrightarrow{T} \Lambda^{p+1} X) / \text{Im}(\Lambda^{p-1} X \rightarrow \Lambda^p X) \quad (p = 0, \dots, n)$$

are called the cohomology groups of $K^\bullet(T, X)$.

Let us suppose that the space

$$H^n(T, X) \cong X / \sum_{i=1}^n T_i X$$

is finite dimensional. Fix a basis $([x_1], \dots, [x_N])$ of this space. Then the spaces $M_k = \sum_{|\alpha|=k} T^\alpha X$ ($k \in \mathbb{N}$) form a decreasing sequence of finite-codimensional subspaces of X such that

$$M_{k+1} = \sum_{j=1}^n T_j M_k \quad (k \in \mathbb{N}).$$

On the algebraic direct sum $\overline{X} = \bigoplus_{k=0}^{\infty} (M_k/M_{k+1})$ we define a commuting tuple $\overline{T} = (\overline{T}_1, \dots, \overline{T}_n)$ of linear maps $\overline{T}_j : \overline{X} \rightarrow \overline{X}$ by

$$\overline{T}_j((x_k + M_{k+1})_{k \in \mathbb{N}}) = (T_j x_{k-1} + M_{k+1})_{k \in \mathbb{N}}.$$

It is elementary to check that the induced module structure

$$\mathbb{C}[z] \times \overline{X} \rightarrow \overline{X}, \quad (p, x) \mapsto p(\overline{T})x$$

turns \overline{X} into a finitely generated graded $\mathbb{C}[z]$ -module. More precisely, let $\mathcal{V}_k = \{p \in \mathbb{C}[z]; \deg(p) \leq k\}$. Then one can show that

$$\bigoplus_{j=0}^{k-1} M_j/M_{j+1} = \text{LH}(\{p(\overline{T})(x_i + M_1); p \in \mathcal{V}_{k-1} \text{ and } i = 1, \dots, N\})$$

for all $k \geq 1$. In particular, one obtains the estimates

$$\dim(X/M_k) \leq N \dim \mathcal{V}_{k-1} = N \frac{k(k+1) \cdot \dots \cdot (k+n-1)}{n!}$$

for all $k \geq 0$. Using a theorem going back to Hilbert (Theorem 1.11 in [5]) we conclude that there is a polynomial $p \in \mathbb{Q}[x]$ with $\deg(p) \leq n$ such that

$$\dim(X/M_k) = p(k)$$

for all sufficiently large natural numbers k . Furthermore, in this case the limit

$$c(T) = n! \lim_{k \rightarrow \infty} \dim(X/M_k)/k^n$$

exists and defines a natural number $c(T) \in \{0, 1, 2, \dots, N\}$. We call p the *Hilbert-Samuel polynomial* and $c(T)$ the *Hilbert-Samuel multiplicity* of T .

The condition that $H^n(T, X)$ is finite dimensional implies that the spaces $H^n(z - T, X) \cong X / \sum_{i=1}^n (z_i - T_i)X$ are finite dimensional for all points z in a suitable open neighbourhood U of 0 in \mathbb{C}^n (Section 2.6 in [4]). For a Banach space E , we denote by \mathcal{O}_U^E the analytic sheaf of all germs of analytic E -valued functions on U . The boundary maps in the Koszul complexes $K^\bullet(z - T, X)$ depend analytically on z and induce a corresponding sequence

$$K^\bullet(z - T, \mathcal{O}_U^X) : 0 \longrightarrow \mathcal{O}_U^{\Lambda^1 X} \xrightarrow{z-T} \dots \xrightarrow{z-T} \mathcal{O}_U^{\Lambda^n X} \longrightarrow 0$$

of analytic sheaves on U . Let us denote by

$$\mathcal{F} = \mathcal{F}_T = \mathcal{H}^n(z - T, \mathcal{O}_U^X) \cong \mathcal{O}_U^X / (z - T) \mathcal{O}_U^{X^n}$$

its last cohomology sheaf. Let $([x_1], \dots, [x_r])$ be a basis of $H^n(T, X)$. After shrinking U one may suppose that there is an epimorphism

$$\mathcal{O}_U^r \xrightarrow{h} \mathcal{F}$$

of analytic sheaves (see the proof of Proposition 9.4.5 in [4]). It follows that the stalk \mathcal{F}_0 of \mathcal{F} at $z = 0$ is a noetherian module over the local ring \mathcal{O}_0 of all convergent power series at $z = 0$. Let \mathfrak{m} be the maximal ideal of \mathcal{O}_0 . It is well known that there is a polynomial $p_{an} \in \mathbb{Q}[x]$ with $\deg(p) \leq n$ such that

$$\dim_{\mathbb{C}}(\mathcal{F}_0 / \mathfrak{m}^k \mathcal{F}_0) = p_{an}(k)$$

for all sufficiently large natural numbers k . Again the limit

$$c_{an}(T) = n! \lim_{k \rightarrow \infty} \dim(\mathcal{F}_0 / \mathfrak{m}^k \mathcal{F}_0) / k^n$$

exists and defines a natural number $c_{an}(T) \in \{0, 1, 2, \dots\}$. We call p_{an} the *analytic Hilbert-Samuel polynomial* and $c_{an}(T)$ the *analytic Hilbert-Samuel multiplicity* of T .

The relationship between the invariants $p, c(T)$ and their analytic counterparts $p_{an}, c_{an}(T)$ was studied before by Douglas and Yan [3], although in a slightly different language. We recall the main results.

Define $M = \{p \in \mathbb{C}[z]; p(0) = 0\}$. Regard X as a $\mathbb{C}[z]$ -module via

$$\mathbb{C}[z] \times X \rightarrow X, \quad (p, x) \mapsto p(T)x.$$

It is elementary to check that $M^k X = \sum_{|\alpha|=k} T^\alpha X$ and that the map

$$\varphi : X \rightarrow \mathcal{F}_0, \quad x \mapsto x + (z - T) \mathcal{O}_0^{X^n}$$

is a morphism of $\mathbb{C}[z]$ -modules such that $\varphi(M^k X) \subset \mathfrak{m}^k \mathcal{F}_0$ for all $k \geq 0$. One can show (cf. Proposition 5 in [3]) that the induced maps

$$\varphi_k : X / M^k X \rightarrow \mathcal{F}_0 / \mathfrak{m}^k \mathcal{F}_0, \quad x + M^k X \mapsto \varphi(x) + \mathfrak{m}^k \mathcal{F}_0$$

are onto for all $k \geq 0$.

Proposition 1 (Douglas-Yan) *Let $T \in L(X)^n$ be a commuting tuple such that $\dim H^n(T, X) < \infty$. Let p and p_{an} be the Hilbert-Samuel polynomial and analytic Hilbert-Samuel polynomial of T , respectively. Then*

$$p_{an}(k) \leq p(k)$$

for all sufficiently large natural numbers k . In particular, the inequality $c_{an}(T) \leq c(T)$ holds.

Let $Z = X/Y$ be the quotient of X modulo a closed invariant subspace Y of T . We denote by $S = T|Y$ and $R = T/Y$ the restriction of T to Y and the quotient tuple of T modulo Y , respectively. The short exact sequences

$$0 \rightarrow K^\bullet(z - S, Y) \rightarrow K^\bullet(z - T, X) \rightarrow K^\bullet(z - R, Z) \rightarrow 0$$

($z \in \mathbb{C}^n$) of Koszul complexes induce long exact sequences of cohomology

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(z-S, Y) & \xrightarrow{j} & H^0(z-T, X) & \xrightarrow{q} & H^0(z-R, Z) \\ & & \downarrow d_z & & \downarrow j & & \downarrow \\ & & H^1(z-S, Y) & \xrightarrow{j} & \dots & & \dots \\ & & \dots & & \dots & & \dots \\ & & \downarrow d_z & & \downarrow j & & \downarrow \\ & & H^n(z-S, Y) & \xrightarrow{j} & H^n(z-T, X) & \xrightarrow{q} & H^n(z-R, Z) \longrightarrow 0, \end{array}$$

where j and q are the linear maps induced by the inclusion $Y \hookrightarrow X$ and the quotient map $X \rightarrow Z$ and the maps d_z are the connecting homomorphisms (cf. [5]).

Our aim is to study the relation between the Hilbert-Samuel multiplicities of T and R . We shall clarify this relation under the additional hypothesis that there is an open neighbourhood U of $0 \in \mathbb{C}^n$ such that all the spaces

$$H^n(z - T, X) \cong X / \sum_{i=1}^n (z_i - T_i)X$$

have the same finite dimension $N \geq 1$. It is well known that the set S of all discontinuity points of the function

$$U \rightarrow \mathbb{N}, \quad z \mapsto \dim H^n(z - R, Z)$$

is a nowhere dense analytic subset of U (Satz 1.5 in [10]) and that there is a natural number $r \in \{0, \dots, N\}$ with

$$\dim H^n(z - R, Z) = r < \dim H^n(w - R, Z)$$

for $z \in U \setminus S$ and $w \in S$.

It is the main aim of this note to prove the following result.

Theorem 2 *Let $R \in L(Z)^n$ be the quotient of a commuting tuple $T \in L(X)^n$ such that*

$$\dim H^n(z - T, X) \equiv \text{const.} < \infty$$

near zero. Then there is nowhere dense analytic subset $S \subset U$ of an open neighbourhood U of $0 \in \mathbb{C}^n$ with the property that

$$c(R) = \dim H^n(z - R, Z) \quad \text{for } z \in U \setminus S.$$

To prove this result we need some preparations. Let U be a connected open neighbourhood of $0 \in \mathbb{C}^n$ such that

$$\dim H^n(z - T, X) = N \quad (z \in U).$$

Choose a direct complement D of $\sum_{i=1}^n T_i X$ in X . Since the analytically parametrized complex

$$T(z) : X^n \oplus D \rightarrow X, \quad ((x_i), y) \mapsto \sum_{i=1}^n (z_i - T_i)x_i + y$$

is onto at $z = 0$, we can achieve (by shrinking U) that the induced map

$$\mathcal{O}(U, X^n \oplus D) \rightarrow \mathcal{O}(U, X)$$

is onto again (Lemma 2.1.5 in [4]). By comparing dimensions we see that the surjective linear maps

$$D \rightarrow H^n(z - T, X), \quad y \mapsto [y]$$

are vector-space isomorphisms for all $z \in U$. Hence, for each $x \in X$ and each $z \in U$, there is a unique vector $x(z) \in D$ with

$$x - x(z) \in \sum_{i=1}^n (z_i - T_i)X.$$

Since $x(z)$ depends analytically on z , we obtain a linear map

$$\rho : X \rightarrow \mathcal{O}(U, D), \quad (\rho x)(z) = x(z).$$

The relation

$$T_j x - z_j x(z) \in (T_j - z_j)x(z) + \sum_{i=1}^n (z_i - T_i)X,$$

valid for $x \in X$, $z \in U$ and $j = 1, \dots, n$, shows that

$$\rho(p(T)x) = p\rho(x) \quad (x \in X, p \in \mathbb{C}[z]).$$

By Lemma 2.1.5 in [4], the morphism

$$\mathcal{O}_U^{X^n \oplus D} \xrightarrow{T(\cdot)} \mathcal{O}_U^X$$

of analytic sheaves remains onto. Therefore the map

$$\mathcal{O}_U^D \rightarrow \mathcal{F}, \quad f_\lambda \mapsto f_\lambda + (z - T)\mathcal{O}_\lambda^{X^n}$$

defines an isomorphism of analytic sheaves.

Let as before $S \subset U$ be a nowhere dense analytic subset such that

$$\dim H^n(z - R, Z) = r$$

for all $z \in U \setminus S$. The above arguments applied to R instead of T show that the sheaf \mathcal{F}_R is locally free of rank r on $U \setminus S$. We call r the *rank* of the sheaf \mathcal{F}_R on U and write $\text{rank}(\mathcal{F}_R) = r$.

Lemma 3 *Let $T \in L(X)^n$ be a commuting tuple such that there is an open neighbourhood U of 0 in \mathbb{C}^n with $\dim H^n(z - T, X) = N$ for $z \in U$. Then we have $c(T) = c_{an}(T) = N$.*

Proof. Denote by $\mathcal{F} = \mathcal{H}^n(z - T, \mathcal{O}_U^X)$ the n -th cohomology sheaf of the complex $K^\bullet(z - T, \mathcal{O}_U^X)$. As seen above, there is an isomorphism $\psi : \mathcal{O}_0^N \longrightarrow \mathcal{F}_0$ of \mathcal{O}_0 -modules. This map induces isomorphisms of \mathcal{O}_0 -modules

$$\psi_k : \mathcal{O}_0^N / \mathfrak{m}^k \mathcal{O}_0^N \longrightarrow \mathcal{F}_0 / \mathfrak{m}^k \mathcal{F}_0 \quad (k \geq 0).$$

Because of $\mathfrak{m}^k \mathcal{O}_0^N = (\mathfrak{m}^k \mathcal{O}_0)^N = (\mathfrak{m}^k)^N$ we obtain induced isomorphisms of \mathcal{O}_0 -modules

$$\mathcal{F}_0 / \mathfrak{m}^k \mathcal{F}_0 \cong \mathcal{O}_0^N / (\mathfrak{m}^k)^N \cong (\mathcal{O}_0 / \mathfrak{m}^k)^N.$$

Using the canonical vector-space isomorphisms $\mathcal{O}_0 / \mathfrak{m}^k \cong \mathcal{V}_{k-1}$, we find that

$$\dim_{\mathbb{C}}(\mathcal{F}_0 / \mathfrak{m}^k \mathcal{F}_0) = \dim(\mathcal{V}_{k-1}^N) = N \frac{k(k+1) \cdot \dots \cdot (k+n-1)}{n!}$$

for all $k \geq 0$. But then $N = c_{an}(T) \leq c(T) \leq N$, and the assertion follows. □

For $z \in U$, define

$$\delta_z : X \rightarrow D, \quad x \mapsto x(z)$$

and

$$Y_z = \{y(z); y \in Y\}.$$

Using the explicit definition of the connecting homomorphisms d_z occurring in the long exact cohomology sequences explained before Theorem 2 one easily checks that the sequences

$$H^{n-1}(z - R, Z) \xrightarrow{d_z} H^n(z - S, Y) \xrightarrow{\widehat{\delta}_z} Y_z \longrightarrow 0$$

are exact for $z \in U$. Using this sequence together with the above long exact cohomology sequence we find that

$$\dim H^n(z - R, Z) = N - \dim Y_z$$

for $z \in U$.

Hence the minimal dimension of $H^n(z - R, Z)$ on U corresponds to the maximal dimension of Y_z on U . The last number admits an algebraic representation (cf. Lemma 4 in [7]). To formulate the relevant result, let us denote by

$$T_k : \mathcal{O}(U, D) \rightarrow \mathcal{O}(U, D), \quad f \mapsto \sum_{|\alpha| \leq k} \frac{f^{(\alpha)}(0)}{\alpha!} z^\alpha$$

the linear maps that associate with each analytic function on U its k -th Taylor polynomial.

Lemma 4 *Let $U \subset \mathbb{C}^n$ be an open neighbourhood of 0 and let D be a finite-dimensional vector space. Given a $\mathbb{C}[z]$ -submodule $M \subset \mathcal{O}(U, D)$, define $M_z = \{f(z); f \in M\}$ for $z \in U$. Then there is a nowhere dense analytic subset $S \subset U$ such that*

$$\dim M_z = \max_{w \in U} \dim M_w = n! \lim_{k \rightarrow \infty} \frac{\dim T_k(M)}{k^n}$$

for all $z \in U \setminus S$.

Proof. It is elementary to check that the proof given in [7] (see Lemma 4) for submodules of Hilbert modules remains valid. For completeness sake, we indicate the main steps.

Fix a basis (e_1, \dots, e_N) of D . Define $m = \max_{w \in U} \dim M_w$ and choose $h_1, \dots, h_m \in M$ such that $h_1(z_0), \dots, h_m(z_0)$ are linearly independent vectors in D for some point $z_0 \in U$. Each function h_j has a representation of the form

$$h_j(z) = \sum_{i=1}^N h_j^i(z) e_i \quad (z \in U)$$

with uniquely determined analytic functions $h_j^i \in \mathcal{O}(U)$. After permuting the given basis of D we may suppose that the analytic matrix-valued function

$$\Theta : U \rightarrow \mathbb{C}^{m,m}, \quad \Theta(\lambda) = (h_j^i(\lambda))_{1 \leq i,j \leq m}$$

is invertible at z_0 . Define $c = \text{ord}_0(\det(\Theta))$. Basic linear algebra allows us to choose an analytic function $A = (A_{ij}) : U \rightarrow \mathbb{C}^{m,m}$ with

$$\Theta(z)A(z) = A(z)\Theta(z) = \det(\Theta(z)) I_m \quad (z \in U).$$

For each $k \in \mathbb{N}$, the analytic functions

$$\Theta_{ij}^k = [\Theta(A - (T_k A_{ij}))]_{i,j} = \sum_{\mu=1}^m h_\mu^i (A_{\mu j} - T_k A_{\mu j})$$

have order at least $k+1$ at $z=0$.

Set $D_0 = \text{LH}\{e_1, \dots, e_m\}$. Let $Q_0 : D \rightarrow D_0$ be the projection onto D_0 with $Q_0 e_i = 0$ for $i = m+1, \dots, N$. Denote by

$$Q = 1 \otimes Q_0 : \mathcal{O}(U, D) \rightarrow \mathcal{O}(U, D_0)$$

the induced projection on $\mathcal{O}(U, D)$. For each function $P = (P_{ij}) : U \rightarrow \mathbb{C}^{m,m}$ with polynomial entries $P_{ij} \in \mathbb{C}[z]$, we have

$$\sum_{i=1}^m [\Theta P]_{ij} e_i = Q \left(\sum_{\mu=1}^m P_{\mu j} h_\mu \right) \in QM$$

for $j = 1, \dots, m$.

Fix $k \in \mathbb{N}$ and a polynomial $p \in \mathbb{C}[z]$. For $j = 1, \dots, m$, the function $T_k(p \det(\Theta) e_j)$ is obtained by applying the linear map T_k coefficientwise to the matrix

$$p \det(\Theta) I_m - p(\Theta_{\mu\nu}^k)_{\mu,\nu} = \Theta p(T_k A_{\mu\nu})_{\mu,\nu},$$

then multiplying the i -th coefficient of the j -th column of this matrix with e_i and adding up over all $i = 1, \dots, m$. It follows that

$$T_k(p \det(\Theta) e_j) \in T_k(QM) \quad (p \in \mathbb{C}[z], j = 1, \dots, m).$$

Let $\mathcal{V}_i = \{p \in \mathbb{C}[z]; \deg(p) \leq i\}$ for $i \in \mathbb{N}$. For $k \geq c$ and $j = 1, \dots, m$, the linear maps

$$\mathcal{V}_{k-c} \rightarrow T_k(QM), \quad p \mapsto T_k(p \det(\Theta) e_j)$$

are obviously injective and therefore

$$\begin{aligned} \dim T_k(M) &\geq \dim Q T_k(M) = \dim T_k(QM) \\ &\geq m \dim \mathcal{V}_{k-c} = \frac{m(k-c+1) \cdot \dots \cdot (k-c+n)}{n!} \end{aligned}$$

We conclude that

$$m \leq n! \liminf_{n \rightarrow \infty} \frac{\dim T_k(M)}{k^n}.$$

Let us turn to the proof of the opposite inequality. Since the right-hand side of the last inequality can be estimated from above against $n! \lim_{k \rightarrow \infty} (N \dim \mathcal{V}_k)/k^n = N$, we may suppose that $m < N$. Let $f = \sum_{i=1}^N f^i e_i \in M$ be arbitrary. The maximality of m implies that the vectors $h_1(z), \dots, h_m(z), f(z) \in D$ are linearly dependent for each $z \in U$. Hence the determinant of the matrix

$$\begin{pmatrix} h_1^1 & \dots & h_m^1 & f^1 \\ \dots & \dots & \dots & \dots \\ h_1^m & \dots & h_m^m & f^m \\ h_1^i & \dots & h_m^i & f^i \end{pmatrix}$$

is identically zero for every fixed $i = m+1, \dots, N$. Expanding this determinant according to the last column, we find that

$$g_1 f^1 + \dots + g_m f^m + g_i f^i \equiv 0$$

with suitable functions $g_1, \dots, g_m \in \mathcal{O}(U)$ and $g_i = \det(\Theta)$.

Fix $k \in \mathbb{N}$. Let $g = T_k(f)$ with $f = \sum_{i=1}^N f^i e_i$ be as above, but assume in addition that $Qg = 0$. Then

$$g = \sum_{i=1}^N T_k(f^i) e_i = \sum_{i=m+1}^N T_k(f^i) e_i,$$

and for $i = m+1, \dots, N$, we obtain the relations

$$T_k(g_i f^i) = -T_k(g_1 f^1 + \dots + g_m f^m) = 0.$$

It follows that $\text{ord}(f^i) \geq k - c + 1$ for $i = m+1, \dots, N$ and that

$$g = (T_k - T_{k-c}) \sum_{i=m+1}^N f^i e_i \in (I - Q)(T_k - T_{k-c})\mathcal{O}(U, D).$$

The above arguments show that

$$\dim(T_k(I - Q)M) \leq \text{rank}(I - Q)(T_k - T_{k-c}) = (N - m) \left[\binom{n+k}{n} - \binom{n+k-c}{n} \right].$$

Since the right-hand side is a polynomial of degree at most $(n-1)$ in k , we conclude that

$$\begin{aligned}
n! \limsup_{k \rightarrow \infty} \frac{\dim T_k(M)}{k^n} &\leq n! \limsup_{k \rightarrow \infty} \frac{\dim Q T_k(M)}{k^n} \\
&\leq n! \lim_{k \rightarrow \infty} \frac{T_k \mathcal{O}(U, D_0)}{k^n} = m.
\end{aligned}$$

□

The significance of the maximal dimension of the spaces M_z , usually referred to as the fibre dimension of M , was recognized before in the context of analytic functional Hilbert spaces by Gleason, Richter and Sundberg [9].

Let us return to the operator-theoretic situation described before Lemma 3. With the notation fixed there, the subspace

$$M = \rho Y \subset \mathcal{O}(U, D)$$

is a $\mathbb{C}[z]$ -submodule. Applying Lemma 4 to this submodule we obtain the next result.

Corollary 5 *With the notation explained above, there is a nowhere dense analytic subset $S \subset U$ of a connected open neighbourhood U of $0 \in \mathbb{C}^n$ such that*

$$\dim H^n(z - R, Z) = \min_{w \in U} H^n(w - R, Z) = N - n! \lim_{k \rightarrow \infty} \frac{\dim T_k(\rho Y)}{k^n}$$

for all $z \in U \setminus S$.

Our next aim is to relate the limit occurring in Corollary 5 to the Hilbert-Samuel multiplicity of R . For a commuting tuple $T \in L(X)^n$, we use the notation $M_k(T) = \sum_{|\alpha|=k} T^\alpha X$ ($k \in \mathbb{N}$).

Lemma 6 *Let $Y \in \text{Lat}(T)$ be a closed invariant subspace of a commuting tuple $T \in L(X)^n$, let $Z = X/Y$ and let $R = T/Y \in L(Z)^n$ be the induced quotient tuple. Suppose that*

$$\dim H^n(T, X) < \infty.$$

Then the Hilbert-Samuel multiplicities of T and R satisfy

$$c(R) = c(T) - n! \lim_{k \rightarrow \infty} \frac{\dim [(Y + M_k(T))/M_k(T)]}{k^n}.$$

Proof. It suffices to observe that the inclusion map $j : Y \rightarrow X$ and the quotient map $q : X \rightarrow Z$ induce short exact sequences

$$0 \longrightarrow \frac{Y + M_k(T)}{M_k(T)} \xrightarrow{j} X/M_k(T) \xrightarrow{q} Z/M_k(R) \longrightarrow 0.$$

Using the fact that the alternating sum of the dimensions of the three spaces forming this sequence is zero, one deduces the assertion. \square

Let us return to the case where $\dim H^n(z - T, X) = N$ near $z = 0 \in \mathbb{C}^n$. By Lemma 3 we know that $c(T) = N$. With the notation fixed before Lemma 4 we obtain that

$$\rho(M_k(T)) \subset \{f \in \mathcal{O}(U, D); \text{ord}_0(f) \geq k\}.$$

Hence the maps $T_{k-1} \circ \rho$ induce surjective linear maps

$$\tau_k : (Y + M_k(T))/M_k(T) \rightarrow T^{k-1}(\rho Y).$$

It follows that the limit occurring in Lemma 6 is at most larger than the corresponding limit in Corollary 5. We complete the proof of Theorem 2 by showing that both limits actually coincide.

Corollary 7 *Let $Y \in \text{Lat}(T)$ be a closed invariant subspace of a commuting tuple $T \in L(X)^n$, let $Z = X/Y$ and let $R = T/Y \in L(Z)^n$ be the induced quotient tuple. Suppose that in some connected open neighbourhood U of $0 \in \mathbb{C}^n$*

$$\dim H^n(z - T, X) = N \quad (z \in U).$$

Then there is a nowhere dense analytic subset $S \subset U$ such that

$$c(R) = \min_{w \in U} \dim H^n(w - R, Z) = \dim H^n(z - R, Z)$$

for $z \in U \setminus S$.

Proof. Let $\mathcal{F} = \mathcal{H}^n(z - T, \mathcal{O}_U^X)$ be the n -th cohomology sheaf of the Koszul complex $K^\bullet(z - T, \mathcal{O}_U^X)$. Fix a direct complement D of $\sum_{i=1}^n T_i X$ in X . As seen before, after shrinking U , we may suppose that the map

$$\mathcal{O}_U^D \rightarrow \mathcal{F}, \quad f_\lambda \mapsto f_\lambda + (z - T)\mathcal{O}_\lambda^{X^n}$$

is an isomorphism of analytic sheaves. The composition

$$X \xrightarrow{\rho} \mathcal{O}(U, D) \cong \mathcal{F}(U) \xrightarrow{\delta_0} \mathcal{F}_0,$$

where the last map is the point evaluation $\delta_0(\gamma) = \gamma(0)$, is precisely the map

$$\varphi : X \rightarrow \mathcal{F}_0, \quad \varphi(x) = x + (z - T)\mathcal{O}_0^{X^n}$$

that we defined in the section leading to Proposition 1. The cited result from [3] implies that the compositions

$$\varphi_k : X/M^k X \xrightarrow{[\rho]} \mathcal{O}(U, D)/M^k \mathcal{O}(U, D) \longrightarrow \mathcal{F}(U)/M^k \mathcal{F}(U) \longrightarrow \mathcal{F}_0/\mathfrak{m}^k \mathcal{F}_0$$

are onto for all $k \geq 0$. Let p and p_{an} be the Hilbert-Samuel polynomial and analytic Hilbert-Samuel polynomial of T , respectively. As an application of Lemma 3 we obtain that $q = p - p_{an}$ is a polynomial with $\deg(q) \leq n - 1$. By construction

$$\dim \text{Ker } \varphi_k = \dim(X/M^k X) - \dim(\mathcal{F}_0/\mathfrak{m}^k \mathcal{F}_0) = q(k)$$

for sufficiently large k . Since τ_k acts as the composition

$$(Y + M_k(T))/M_k(T) \xrightarrow{[\rho]} \mathcal{O}(U, D)/M^k \mathcal{O}(U, D) \xrightarrow{\hat{T}_{k-1}} \mathcal{O}(U, D),$$

where the map \hat{T}_{k-1} defined by $\hat{T}_{k-1}([f]) = T_{k-1}(f)$ is injective, we conclude that

$$\dim \text{Ker } \tau_k \leq \dim \text{Ker } \varphi_k \quad (k \in \mathbb{N}).$$

The observation that

$$\dim [(Y + M_k(T))/M_k(T)] - \dim T_{k-1}(\rho Y) = \dim(\text{Ker } \tau_k) \leq q(k)$$

for all sufficiently large k , completes the proof. □

The question whether $c(R) = c_{an}(R)$ in the setting of Corollary 7 remains open here, at least in the Banach-space case. In the case of Hilbert spaces the cohomology sheaf $\mathcal{H} = \mathcal{H}^n(z - R, \mathcal{O}_U^Z)$ is known to be coherent [11]. Then standard results from analytic geometry (Theorem 7.4 in [2]) imply that the map

$$U \rightarrow \mathbb{N}, \quad z \mapsto c_{an}(z - R) = n! \lim_{k \rightarrow \infty} \frac{\dim(\mathcal{H}_z/\mathfrak{m}_z^k \mathcal{H}_z)}{k^n}$$

is upper semicontinuous. Since

$$c_{an}(z - R) = \min_{w \in U} \dim H^n(w - R, Z) = c(R) \quad (z \in U \setminus S)$$

for a proper analytic subset $S \subset U$, it follows that $c_{an}(R) \geq c(R)$. Since the reverse inequality always holds, we have equality. The sheaf \mathcal{H} is also known to be coherent, when the tuple R is Fredholm, that is, when $\dim H^p(R, Z) < \infty$ for $p = 0, \dots, n$. Hence also in this case we obtain equality $c(R) = c_{an}(R)$ in the setting of Corollary 7.

A second natural question is whether the assertion of Theorem 2 remains true, when we replace the hypothesis that R is a quotient of a tuple $T \in L(X)^n$ for which $\dim H^n(z - T, X)$ is constant for z near 0 simply by the condition that $\dim H^n(R, Z) < \infty$.

References

- [1] W. Arveson, The curvature invariant of a Hilbert module over $\mathbb{C}[z_1, \dots, z_d]$, J. Reine Angew. Math. 522 (2000), 173-236.
- [2] E. Bierstone and P. Milman, Relations among analytic functions I, Ann. Inst. Fourier 37 (1987), 187-239.
- [3] R. Douglas and K. Yan, Hilbert-Samuel polynomials for Hilbert modules, Indiana Univ. Math. J. 42 (1993), 811-820.
- [4] J. Eschmeier and M. Putinar, Spectral decompositions and analytic sheaves, London Mathematical Society Monographs, New Series, 10, Clarendon Press, Oxford, 1996.
- [5] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, Graduate texts in Mathematics 150, Springer-Verlag, New York, 1995.
- [6] X. Fang, Samuel multiplicity and the structure of semi-Fredholm operators, Adv. Math. 186 (2004), 411-437.
- [7] X. Fang, The Fredholm index of quotient Hilbert modules, Math. Res. Lett. 12 (2005), 911-920.
- [8] X. Fang, The Fredholm index of a pair of commuting operators, Preprint.
- [9] J. Gleason, S. Richter, and C. Sundberg, On the index of invariant subspaces in spaces of analytic functions in several complex variables, Crelles Journal, to appear.
- [10] W. Kabbalo, Holomorphe Semi-Fredholmfunktionen ohne komplementierte Bilder, Math. Nachr. 91 (1979), 327-335.
- [11] A. Markoe, Analytic families of differential complexes, J. Funct. Anal. 9 (1972), 181-188.

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